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Reconstruction of controls in exponentially stable linear systems subjected to small perturbations☆

V.I. Maksimov

Ekaterinburg, Russia

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Abstract

The problem of the dynamical reconstruction of the variable input of an exponentially stable linear system subjected to small non-linear perturbations is considered. In the case of inaccurate observations of its phase trajectory, an algorithm for solving this problem is given, based on the method of control with a model. The algorithm is stable to data interference and computation errors. General constructions are illustrated by an example in which the problem of reconstructing the input of an oscillatory section is discussed.

1. Introduction. Formulation of the problem

We will consider a controlled system described by the differential equation

$$\begin{aligned} &\dot{y}(t) = Ay(t) + f(y(t)) + Cu(t) \\ &t \in T = [0, +\infty), \quad y(0) = y_0; \quad y \in R^n; \quad u \in R^q \end{aligned} \tag{1.1}$$

where C is an $(n \times q)$-dimensional matrix, f is an $(n \times n)$-dimensional matrix function satisfying the Lipschitz condition

$$|f(x) - f(y)| \le L|x - y|, \quad \forall x, y \in R^n$$

and $|x|$ is the Euclidean norm of the vector x. The trajectory of the system

$$y(t) = \{y_1(t), y_2(t)\}, \quad y_1 \in R^{n_1}, \quad n_1 < n, \quad y_2 \in R^{n - n_1}$$

depends on the time-variable input action (control) $u = u(t)$. Neither this control nor the trajectory is specified in advance. During motion, some signal characterizing the phase state of the system is observed. A part of the coordinates of system (1.1) – the coordinates $y_2(\tau_i)$ – is measured with an error at discrete, fairly frequent instants of time $\tau_i \in T$ ($i = 1, 2, \ldots$). The results of the measurements – the vectors $\xi_i^h \in R^{n-n_1}$ – are such that

$$\xi_i^h = y_2(\tau_i) + z_i, \quad |z_i| \le \nu_i^h$$

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E-mail address: maksimov@imm.uran.ru.

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Here $v_i^h \in (0, 1)$ is the measurement error at the time τ_i, and the number $h \in (0.1)$ characterizes the accuracy of the measurement.

The problem discussed in this paper is as follows. Along with the controlled system (1.1), there is one other dynamical system, which we will call the reference system, that is subjected to an uncontrolled input action v. The trajectory of this system, and also the input action, are unknown *a priori*. However, it is possible at instants of time τ_i to calculate (with an error) all the coordinates of the reference system (or part of them). It is required to construct a law of the formation of the control u, based in the feedback principle, that ensures closeness of the trajectories of the available systems. Here, the control u, formed when developing the process, must approximate, in the root mean square to the unknown input action v of the reference system.

Suppose the reference motion is described by the system

$$\dot{x}(t) = Ax(t) + f(x(t)) + Cv(t), \quad t \in T, \quad x(0) = x_0 \tag{1.2}$$

It is assumed that both the function $v(t)$ (the input action) and the solution $x(t) = x(t; x_0; v(\cdot))$ of this system are also unknown. All that is known in that the function $v(t)$ is constrained:

$$v(\cdot) \in Q(\cdot) = \{v(t), t \in T\colon v(t) \in Q \text{ when a.e. } t \in T\} \tag{1.3}$$

where $Q \in R^q$ is a specified convex, bound and closed set. At instants of time τ_i ($i \geq 1$), the state is measured (with an error):

$$x_2(\tau_i) \in R^{n-n_1} \quad (x(t) = \{x_1(t), x_2(t)\})$$

The results of measurements – the vectors $\psi_i^h \in R^{n-n_1}$ – satisfy the inequalities

$$\left|\psi_i^h - x_2(\tau_i)\right| \leq \nu_i^h \tag{1.4}$$

It is required to develop an algorithm of the formation, according to the feedback principle, of the control $u = u^h(\tau_i, \xi_i^h, \psi_i^h)$, $t \in [\tau_i, \tau_{i+1})$, in system (1.1) such that, first, the trajectory of system (1.1) that corresponds to this control ($y^h(\cdot) = y(\cdot; y_0, u^h(\cdot))$ is retained at all $t \in T$ in a certain fairly close vicinity of the solution of reference system (1.2), i.e.

$$\sup_{t \in T}\left|x(t) - y^h(t)\right| \leq \nu(h), \quad \nu(h) \to 0 \text{ as } h \to 0 \tag{1.5}$$

and, second, the control $u = u^h(\cdot)$ approximates, in the root-mean-square, the unknown input $v(\cdot)$ i any finite time interval, i.e.

$$\varphi(h, \vartheta) \equiv \|v(\cdot) - u^h(\cdot)\|_{L_2([0, \vartheta]; R^q)} = \left(\int_0^\vartheta \left|u^h(\tau) - v(\tau)\right|^2 d\tau\right)^{1/2} \to 0$$
$$\text{as } h \to 0, \quad \forall \vartheta \in T \tag{1.6}$$

These are the essentials of the problem being considered in this paper.

The problem belongs to the class of inverse problems of the dynamical estimation of unknown characteristics from the results of measurements. (Problems of this kind were investigated, for example, in Refs 1–4. In particular, it can be solved using dynamical inversion theory[5–10] (here, only monographs and review papers in which appropriate references can be found are mentioned). We will bear in mind the fact that the algorithms proposed in these papers for solving corresponding problems of reconstruction are oriented towards a finite functioning time interval of the system $T = [0, v]$, $v < +\infty$. Note also that, as v increases, when implementing the algorithms described,[5–10] "accumulation" of computation and measurement errors occurs. As v increases the rates of convergence of the algorithms deteriorate. Thus, the quality of the algorithms, generally speaking, depends on the length of the time interval for which the system is functioning. Below, an algorithm for solving the problem will be given that is "independent" of the length of this interval. We will consider the case where the matrix A is exponentially stable, while the non-linear part f of system (1.1) comprises a perturbation that is suitably consistent with A. The algorithm is based on ideas set out in earlier studies.[5,6,10,11]

2. Auxiliary constructions

Before describing of the reconstruction algorithm, we will develop some necessary auxiliary constructions. Below, we will assume that

$$x_0 = \xi_0^h, \quad \left|\xi_0^h - y_0\right| \le \nu_0^h$$

Suppose the following condition is satisfied.

Condition 1. Matrix A is stable.

We will fix some symmetrical positive–definite $(n \times n)$-dimensional matrix Q. As is well known, a unique $(n \times n)$-dimensional positive-definite symmetrical matrix D there exists such that the Lyapunov function $V(x) = x'Dx$ satisfies the equation

$$(\partial V/\partial x)'Ax + x'Qx = 0, \quad \forall x \in R^n \tag{2.1}$$

(the prime denotes transposition). Note also that it is not difficult to specify a number $c > 0$, for which the following inequality is satisfied

$$x'Qx \ge c|x|^2, \quad \forall x \in R^n \tag{2.2}$$

We will fix the family of subdivisions of interval T

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{\infty}, \quad \tau_{h,0} = 0, \quad \tau_{h,i+1} = \tau_{h,i} + \delta_i(h) \tag{2.3}$$

Let $X_T(\cdot)$ be a bundle of solutions of system (1.2), i.e.

$$X_T(\cdot) = \{x(\cdot) : x(\cdot) = x(\cdot; x_0, \upsilon(\cdot)), \upsilon(\cdot) \in Q(\cdot)\}$$

We will assume that the bundle $X_T(\cdot)$ is bounded, i.e. all the solutions of system (1.2) $x = x(t)$ remain in some bounded region H^X of phase space R^n:

$$x(t) \in H^X, \quad \forall t \in T, \quad \forall x(\cdot) \in X_T(\cdot) \tag{2.4}$$

Let $z(t) = y^h(t) - x(t)$, where $x(t)$ is the solution of system (1.2) and $y^h(t)$ is the solution of system (1.1), corresponding to the control

$$u = u^h(\cdot) \in Q(\cdot) : y^h(\cdot) = y(\cdot; y_0, u^h(\cdot))$$

Then, the function $z(t)$ satisfies the vector equation

$$\dot{z}(t) = Az(t) + C(u^h(t) - \upsilon(t)) + \tilde{f}(t) \text{ when a.e. } t \in T; \tilde{f}(t) = f(y^h(t)) - f(x(t)) \tag{2.5}$$

with the initial condition $z(0)$.

The symbol $\dot{V} = (t)|_{(2.5)}$ will denote the derivative of the Lyapunov function $V(x) = x'Dx$, calculated by virtue of system (2.5). Thus

$$\dot{V}(t)|_{(2.5)} = (\partial V/\partial x|_{x = z(t)})'(Az(t) + C(u^h(t) - \upsilon(t)) + \tilde{f}(t)) \text{ when a.e. } t \in T$$

Lemma 1. *Suppose Condition 1 is satisfied, and also the inequality $c > 2L|D|$. Then the following inequality holds*:

$$\dot{V}(t)|_{(2.5)} \le 2(Dz(t))'C(u^h(t) - \upsilon(t)) \text{ when a.e. } t \in T$$

Here and below, the norm of the matrix means its Euclidean norm.

Proof. Using relations (2.1) and (2.2), we will have

$$\dot V(t)|_{(2.5)} = -z'(t)Qz(t) + (\partial V/\partial x|_{x=z(t)})'(C(u^h(t)) - v(t)) + \tilde f(t)) \le \\ \le (\partial V/\partial x|_{x=z(t)})'(C(u^h(t) - v(t)) + \tilde f(t)) - c|z(t)|^2 \text{ when a.e. } t \in T \quad (2.6)$$

Further, since the Lipschitz condition is satisfied for the function f, the following inequality will hold:

$$(\partial V/\partial x|_{x=z(t)})'\tilde f(t) = 2(Dz(t))'\tilde f(t) \le 2L|D||z(t)|^2 \text{ when a.e. } t \in T \quad (2.7)$$

From relations (2.6) and (2.7) we obtain the estimate

$$\dot V(t)|_{(2.5)} \le 2(Dz(t))'C(u^h(t) - v(t)) + (2L|D| - c)|z(t)|^2 \text{ when a.e. } t \in T$$

from which the assertion of the lemma follows. □

3. An algorithm for the solution. The case when all the coordinates are measured

We will write an algorithm for solving the problem being considered, to beginning with the case when all the coordinates are measured. We fix the function

$$\alpha(h) : (0, 1) \to R^+ = \{r \in R : r > 0\}$$

(regularizer) and the family of subdivisions Δ_h (2.3). We select the latter in such a way that the following condition is satisfied.

Condition 2. The family Δ_h and the errors of measurements ν_i^h are such that we have the following relations

$$\sum_{i=0}^{+\infty} \delta_i(h) = +\infty, \quad \forall h \in (0, 1), \quad \nu_0^h \le \psi(h) \to 0+$$

$$\sum_{i=0}^{+\infty} \delta_i(h)(\nu_i^h + \delta_i(h)) \le \varphi(h) \to 0+ \text{ as } h \to 0+$$

Remark. Condition 2 is satisfied, for example, if

$$\delta_i(h) = \nu_i^h = dh/(i+1)^\gamma, \quad \gamma \in (0.5; 1], \quad i = 0, 1, \dots, \quad d = \text{const} > 0$$

Here

$$\varphi(h) = 2h^2d^2 \sum_{i=1}^{\infty} i^{-2\gamma}, \quad \psi(h) = h$$

Inequalities (1.4) take the form

$$\left|\psi i^b - x(\tau_i)\right| \le dh/(i+1)$$

Before the start of the operation of the algorithm, we fix the value of h, the family $\{\nu_i^h\}_{i=0}^{\infty}$, the quantity $\alpha = \alpha(h)$ and the subdivision $\Delta_h = \{\tau_{h,i}\}_{i=0}^{\infty}$. We divide the operation of the algorithm into steps of the same type. Suppose that, during the i-th step, carried out in the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following operations are carried out. First, at time τ_i, the vector u_i^h is calculated by means of the formula

$$u_i^h = \arg\min\{2(D(\psi_i^h - \xi_i^h))'Cv + \alpha|v|^2 : v \in Q\} \quad (3.1)$$

Then, the following control is fed to the input of system (1.1) at all $t \in \delta_i$

$$u = u^h(t) = u_i^h \quad (3.2)$$

As a result, under the action of this control and some unknown perturbation $v(t)$, $t \in \delta_i$, system (1.1) transfers from the state $y^h(\tau_i)$ to the state $y^h(\tau_i+1)$, while reference system (1.2) transfers from the state $x(\tau_i)$ to the state $x(\tau_i+1)$. At the next, $(i+1)$-th step, similar actions are repeated.

The symbol C^+ denotes a pseudoinverse matrix.

Theorem 1. *Let* $\alpha(h) \to 0$, $(\psi^2(h)+\varphi(h))\alpha^{-1}(h) \to 0$ *at* $h \to 0$ *and* $v_*(t) = C^+(\dot{x}(t) - Ax(t) - f(x(t)))$, *and* Conditions 1 and 2 *be satisfied, as well as the inequality* $c > 2L|D|$. *Then, whatever the number* $v \in T$, *the following convergence occurs*

$$u^h(\cdot) \to v_*(\cdot) \text{ in } L_2([0, \vartheta]; R^q) \text{ as } h \to 0$$

The theorem is proved in the same way as indicated earlier[10] and is based on Lemma 2 given below.

Note that the method proposed by Kryazhimskii and Osipov[10] and developed subsequently[5–9] is based on the idea of stabilizing suitable Lyapunov-type functionals by an extremal shear. Thus, the method combines the stabilization principle with the external shift rule. In the case being considered, a regularized extremal shear is defined by relation (3.1).

Lemma 2. *Suppose the conditions of* Theorem 1 *are satisfied. Then the following inequalities hold*:

$$\left|x(\tau_i) - y^h(\tau_i)\right|^2 \le C_1(\psi^2(h)+\varphi(h)) + \alpha(h)\int_0^{\tau_1} |v_*(\tau)|^2 d\tau$$
$$\int_0^{\tau_i} \left|u^h(\tau)\right|^2 d\tau \le \int_0^{\tau_i} |v_*(\tau)|^2 d\tau + C_2\frac{\psi^2(h)+\varphi(h)}{\alpha(h)}, \quad \forall \tau_i = \tau_{h,i} \in \Delta_h \tag{3.3}$$

The constants C_1 *and* C_2 *are in dependent of* $x(\cdot)$, $y^h(\cdot)$ *and* i.

Proof. The proof of the lemma is based on the procedure of stabilizing the Lyapunov-type functional

$$\varepsilon(t) = V(z(t)) + \alpha(h)\int_0^t \{\left|u^h(s)\right|^2 - |v_*(s)|^2\}ds; \quad z(t) = y^h(t) - x(t) \tag{3.4}$$

Using Lemma 1, we conclude that

$$\dot{\varepsilon}(t) \le 2(Dz(t))'C(u^h(t) - v_*(t)) + \alpha(h)\{\left|u^h(t)\right|^2 - |v_*(t)|^2\} \text{ when a.e. } t \in T$$

By virtue of including expressions (2.4) evenly for all $x(\cdot) \in X^T(\cdot)$ and $y^h(\cdot)$

$$|z(t) - z(\tau_i)| \le b_1(t - \tau_i), \quad t \in \delta_i = \delta_{i,h} = [\tau_i, \tau_{i+1}), \quad \tau_i = \tau_{i,h}$$

where the constant b_1 is independent of $x(\cdot)$, $y^h(\cdot)$, t and i. In such a case

$$\dot{\varepsilon}(t) \le 2(D(\psi_i^h - \xi_i^h))'C(u^h(t) - v_*(t)) +$$
$$+ \alpha(h)\{\left|u^h(t)\right|^2 - |v_*(t)|^2\} + b_2(v_i^h + \delta_i(h)) \text{ when a.e. } t \in \delta_i$$

Taking into account the rule for determining the control $u^h(\cdot)$ (see relations (3.1) and (3.2)), we obtain

$$\dot{\varepsilon}(t) \le b_2(v_i^h + \delta_i(h)) \text{ when a.e. } t \in \delta_i$$

Thus the following inequality holds

$$\varepsilon(\tau_{i+1}) \le \varepsilon(\tau_i) + \delta_i(h)b_2(v_i^h + \delta_i(h))$$

from which, by virtue of Condition 2, we obtain the inequality

$$\varepsilon(\tau_i) \le (\nu_0^h)^2 + b_2 \sum_{j=0}^{i-1} \delta_i(h)(\nu_i^h + \delta_i(h)) \le \psi^2(h) + b_2\varphi(h)$$

The of lemma is proved. □

Proof of Theorem 1. We will show that, for an arbitrary sequence $h_j \to 0+$ as $j \to \infty$, for any number $\upsilon \in T$, for any family $\{\Delta_{h_j}\}$ of subdivisions of the interval T with diameters $\delta(h_j)$, such that

$$\{\psi^2(h_j) + \varphi(h_j)\}/\alpha(h_j) \to 0 \text{ as } j \to \infty$$

and for any measurements of $\xi_i^{h_j}$ and $\psi_i^{h_j}(|\xi_i^{h_j} - y(\tau_i)| \le \nu_i^{h_j},\ |\psi_i^{h_j} - x(\tau_i)| \le \nu_i^{h_j})$, the following convergence occurs

$$u^{h_j}(\cdot) \to \upsilon_*(\cdot) = \upsilon_*(\cdot; x(\cdot)) \text{ в } L_2 \text{ as } j \to \infty \tag{3.5}$$

Here and below, $L_2 = L_2(T_*; R^q)$, $T_* = [0, \upsilon]$ and the controls $u^{h_j}(\cdot)$ are defined by the rule (3.1), (3.2).

Assuming the opposite, we conclude that a sub-sequence of the sequence $u^{h_j}(\cdot)$ will be found (we will denote it, for simplicity, by the same notation $u^{h_j}(\cdot)$) such that

$$u^{h_j}(\cdot) \to u_0(\cdot) \text{ when in } L_2 \text{ as } j \to \infty\ u_0(\cdot) \ne \upsilon_*(\cdot; x(\cdot)) \tag{3.6}$$

In such a case, choosing if necessary, a sub-sequence again from h_j, we assume

$$y^{h_j}(\cdot) \to y_*(\cdot) \text{ in } C(T_*; R^n) \text{ as } j \to \infty$$

where $y^{h_j}(\cdot) = y(\cdot; y_0, u^{h_j}(\cdot))$ and $y_*(\cdot)$ is the solution of the equation

$$\dot{y}(t) = Ay(t) + Cu_0(t) + f(y(t)), \quad t \in T_*, \quad y(0) = y_0$$

By virtue of Lemma 2 (see the first inequality of system (3.3)) we have

$$y_*(t) = x(t), \quad t \in T_*$$

This means that $u_0(\cdot) \in U_\upsilon(x(\cdot))$ and consequently

$$|u_0(\cdot)|_{L_2} \ge |\upsilon_*(\cdot; x(\cdot))|_{L_2} \tag{3.7}$$

The notation $U_\upsilon(x(\cdot))$ denotes the set of all functions compatible with the output $x(t)$, $t \in [0, \upsilon]$, i.e.

$$U_\vartheta(x(\cdot)) = \{u(\cdot) \in L_2([0, \vartheta]; R^q) : Cu(t) =$$
$$= \dot{x}(t) - Ax(t) - f(x(t)) \text{ when a.e. } t \in [0, \vartheta]\}$$

Furthermore, by virtue of the known properties of the weak limit, from relation (3.6) we have the inequality

$$\underline{\lim}_{j \to \infty} |u^{h_j}(\cdot)|_{L_2} \ge |u_0(\cdot)|_{L_2} \tag{3.8}$$

In turn, from Lemma 2 (see the second inequality of system (3.3)) we likewise have the inequality

$$|u^{h_j}(\cdot)|_{L_2}^2 \le |\upsilon_*(\cdot; x(\cdot))|_{L_2}^2 + C_2\{\psi^2(h_j) + \varphi(h_j)\}/\alpha(h_j) \tag{3.9}$$

by virtue of which we have

$$\overline{\lim}_{j \to \infty} |u^{h_j}(\cdot)|_{L_2} \le |\upsilon_*(\cdot; x(\cdot))|_{L_2}$$

i.e. (see inequalities (3.7) and (3.8))

$$\overline{\lim_{j\to\infty}}|u^{h_j}(\cdot)|_{L_2} \le |v_*(\cdot;\, x(\cdot))|_{L_2} \le |u_0(\cdot)|_{L_2} \le \underline{\lim_{j\to\infty}}|u^{h_j}(\cdot)|_{L_2} \quad (3.10)$$

Since the set $U_v(x(\cdot))$ contains a unique element of the minimal L_2-norm (namely $v_*(\cdot; x(\cdot))$), from inequalities (3.10) we obtain

$$u_0(\cdot) = v_*(\cdot) \quad (3.11)$$

Using relations (3.6) and (3.11), we conclude that

$$u^{h_j}(\cdot) \to v_*(\cdot;\, x(\cdot)) \text{ weakly in } L_2 \text{ as } j \to \infty \quad (3.12)$$

The convergence (3.12) contradicts relation (3.6) and the assumption $u_0(\cdot) \neq v_*(\cdot; x(\cdot))$.

The theorem is proved. □

Remarks. 1°. Suppose system (1.1) is non-linear in the phase variable, i.e. $Ax = A(x)$, where $A{:}R^n \to R^n$ is a non-linear Lipschitz function. Likewise, suppose a function $V{:}R^n \to R$ exists with the following properties:

(a) it is possible to specify the constant $c > 0$ such that

$$(\partial V/\partial z)|_{z=x-y}(A(x) - A(y)) \le -c|z|^2, \quad \forall x, y \in R^n$$

(b) $V(z) \ge w(|z|)$, $\forall z \in R^n$, where $w : R \to R$ is a continuous function with the properties: $w(0) = 0$, $w(r) > 0$ when $r \neq 0$. In this case, Theorem 1 will remain true if the vectors u_i^h are calculated by means of the formula

$$u_i^h = \arg\min\left\{((\partial V/\partial z)|_{z=\psi_i^h} - (\partial V/\partial z)|_{z=\xi_i^h})'Cv + \alpha|v|^2 : v \in Q\right\}$$

2°. As can be seen from the Lemma 2 proof, by selecting the control $u = u^h(t)$ according to formulae (3.1) and (3.2), we ensure a "small" increase in the Lyapunovs functional $\varepsilon(t)$. In turn, the proof of Theorem 1 implies that the functional $\varepsilon(t)$ is chosen such that, from the smallness of its values, the "closeness" of $u^h(\cdot)$ to $v_*(\cdot)$ follows.

4. A solution algorithm. The case when some of the coordinates are measured

We will now consider the case when some of the coordinates are measured. Suppose that $n > q$ and the matrix C has the following structure: $C = [O, C_1]$, where O is an $n_1 \times q$ null matrix and C_1 is an $(n - n_1) \times q$ matrix. Then, system (1.1) can be written in the form (δ_{2j} is the Kronecker delta, $j = 1, 2$ in all cases below)

$$\dot{y}_j(t) = A_{2j-1}y_1(t) + A_{2j}y_2(t) + f_j(y_1(t), y_2(t)) + \delta_{2j}C_1u(t) \quad (4.1)$$

In turn, reference system (1.2) will take the form

$$\dot{x}_j(t) = A_{2j-1}x_1(t) + A_{2j}x_2(t) + f_j(x_1(t), x_2(t)) + \delta_{2j}C_1v(t) \quad (4.2)$$

As above, we consider the functions f_j to be Lipschitz functions with Lipschitz constants L_j respectively, i.e.

$$|f_j(x_1, y_1) - f_j(x_2, y_2)| \le L_j\{|x_1 - x_2| + |y_1 - y_2|\}, \quad \forall (x_1, x_2), (y_1, y_2) \in R^n$$

Suppose the following condition is satisfied.

Condition 3. Matrices A_1 and A_4 are stable.

We fix symmetric positive definite matrices D_1 and D_2 of dimensions $(n_1 \times n_1)$ and $(n-n_1)\times(n-n_1)$ respectively. Uniquely defined symmetrical positive definite matrices Q_1 and Q_2 correspond to these matrices such that

$$(\partial V_1/\partial x_1)'A_1x_1 + x_1'Q_1x_1 = 0, \quad \forall x_1 \in R^{n_1} \tag{4.3}$$

$$(\partial V_2/\partial x_2)'A_4x_2 + x_2'Q_2x_2 = 0, \quad \forall x_2 \in R^{n-n_1} \tag{4.4}$$

where the Lyapunov functions V_j are such that $V_j = x_j'D_jx_j$. Let

$$x_j'Q_jx_j \ge c_j|x_j|^2, \quad \forall x_j \in R^{(j-1)n+(3-2j)n_1} \tag{4.5}$$

We will introduce the notation

$$\dot V_j(t)\big|_{(4.1)} = (\partial V_j/\partial x|_{x=z_j(t)})'(A_{2j-1}z(t) + A_{2j}z(t) + \Delta f_j + \delta_{2j}C_1(\upsilon(t)-u^h(t)))$$

$$\Delta f_j = f_j(x_1(t), x_2(t)) - f_j(y_1^h(t), y_2^h(t))$$

$$\dot V(t)\big|_{(4.1)} = \dot V_1(t)\big|_{(4.1)} + \dot V_2(t)\big|_{(4.1)}$$

where $z(t) = x(t) - y^h(t)$, $x(t) = x(t; x_0, \upsilon(\cdot)) = (x_1(t), x_2(t))$ is the solution of system (4.2) and $y^h(t) = y(t; y_0, u(\cdot)) = (y_1^h(t), y_2^h(t))$ is the solution of system (4.1), which corresponds to the control $u = u^h(\cdot) \in Q(\cdot)$. As above, we will assume that the bundle of solutions of systems (1.2) ((4.2)) is bounded, i.e. relation (2.4) holds.

We will introduce the following condition.

Condition 4. The following inequalities hold

$$2L_j|D_1| + ((L_1 + |A_2|)|D_1|^2) + ((L_2 + |A_3|)|D_2|)^2 \le c_j \tag{4.6}$$

Lemma 3. *Suppose* Conditions 3 and 4 *are satisfied. Then the following inequality holds*

$$\dot V(t)\big|_{(4.1)} \le 2(D_2z_2(t))'C_1(\upsilon(t) - u^h(t)) \text{ when a.e. } t \in T$$

Proof. Using relations (4.3) and (4.5) when $j = 1$, we will have

$$\begin{aligned}\dot V_1(t)\big|_{(4.1)} &= -z_1'(t)Q_1z_1(t) + (\partial V_1/\partial x|_{x=z_1(t)})'(A_2z_2(t) + \Delta f_1) \le \\ &\le (\partial V_1/\partial x|_{x=z_1(t)})'(A_2z_2(t)) + \Delta f_1) - c_1|z_1(t)|^2 \text{ when a.e. } t \in T\end{aligned} \tag{4.7}$$

Further, by virtue of the Lipschitz nature of the function f_1, the following inequality holds

$$\begin{aligned}(\partial V_1/\partial x|_{x=z_1(t)})'(A_2z_2(t) + \Delta f_1) &= 2(D_1z_1(t))'(A_2z_2(t) + \Delta f_1) \le \\ &\le 2L_1|D_1|(|z_1(t)|^2 + |z_1(t)||z_2(t)|) + 2|D_1||A_2||z_1(t)||z_2(t)| \text{ when a.e. } t \in T\end{aligned} \tag{4.8}$$

From inequalities (4.7) and (4.8) we obtain the estimate

$$\dot V_1(t)\big|_{(4.1)} \le (2L_1|D_1| - c_1)|z_1(t)|^2 + 2(L_1 + |A_2|)|D_1||z_1(t)||z_2(t)| \text{ when a.e. } t \in T \tag{4.9}$$

Similarly, we derive

$$\begin{aligned}\dot V_2(t)\big|_{(4.1)} &\le (2L_2|D_2| - c_2)|z_2(t)|^2 + \\ &+ 2(L_2 + |A_3|)|D_2||z_1(t)||z_2(t)| + 2(D_2z_2(t))'C_1(\upsilon(t) - u^h(t))\end{aligned} \tag{4.10}$$

The assertion of the lemma follows from relations (4.9) and (4.10) and Condition 4.

We will now describe the algorithm for solving the problem in the case when $y_2(\tau_i)$ and also $x_2(\tau_i)$ are measured. Suppose the function $\alpha = \alpha(h) : (0, 1) \to R^+$ and the family of subdivisions $\Delta_{\underline{h}}$ (2.3) have been selected. Before the algorithm starts to operate, we fix the value of h, the family $\{\nu_i^h\}_{i=0}^{\infty}$, the subdivision $\Delta_h = \{\tau_{h,i}\}_{i=0}^{\infty}$ and the quantity $\alpha = \alpha(h)$.

We will divide the operation of the algorithm into steps of the same type. In the course of the i-th step, performed in the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following operations are carried out. First, at time τ_i, the vector u_i^h is calculated by means of the formula

$$u_i^h = \arg\min\{2(D_2(\psi_i^h - \xi_i^h))' C_1 \upsilon + \alpha|\upsilon|^2 : \upsilon \in Q\} \tag{4.11}$$

where

$$\psi_i^h, \xi_i^h \in R^{n-n_1}, \quad |\psi_i^h - x_2(\tau_i)| \le \nu_i^h, \quad |\xi_i^h - y_2(\tau_i)| \le \nu_i^h$$

Then, to the input of system (4.1), at all $t \in \delta_i$, the following control is applied

$$u = u^h(t) = u_i^h \tag{4.12}$$

As a result, under the action of this control and some unknown perturbation $u(t)$, $t \in \delta_i$, system (4.1) transfers from the state $y^h(\tau_i)$ to the state $y^h(\tau_{i+1})$, and the reference system (4.2) transfers from the state $x(\tau_i)$ to the state $x(\tau_{i+1})$. At the next, $(i+1)$-th step, similar actions are repeated. □

Theorem 2. *Suppose*

$$\alpha(h) \to 0, \quad (\psi^2(h) + \varphi(h))\alpha^{-1}(h) \to 0 \text{ as } h \to 0$$

$$\upsilon_*(t) = C_1^+(\dot{x}_2(t) - A_3 x_1(t) - A_4 x_2(t) - f_2(x_1(t), x_2(t)))$$

and Conditions 2 to 4 are satisfied. Then, whatever the number $\upsilon \in T$, the following convergence occurs

$$u^h(\cdot) \to \upsilon_*(\cdot) \text{ in } L_2([0, \vartheta]; R^q) \text{ as } h \to 0$$

The proof is similar to the proof of Theorem 1. It is based on the procedure for stabilizing the Lyapunov-type functional which differs from functional (3.4) in that $V(z(t))$ is replaced by $V_1(z_1(t)) + V_2(z_2(t))$.

5. Example

As a model example illustrating the algorithm given above, we will consider an oscillatory section described by the equation

$$\begin{aligned} &\ddot{w}(t) + 2l\dot{w}(t) + \omega_0^2 w(t) = u(t), \quad t \in [0, +\infty); \quad l = \text{const} > 0, \quad \omega_0 \ne 0 \\ &w(0) = w_0, \quad \dot{w}(0) = w_1 \end{aligned} \tag{5.1}$$

Assuming that $y_1 = w$ and $y_2 = \dot{w}$, we will change to the system

$$\dot{y}_1(t) = y_2(t), \quad \dot{y}_2(t) = -\omega_0^2 y_1(t) - 2l y_2(t) + u(t)$$

Considering that

$$D = \left\| \begin{matrix} 2l^2 + \omega_0^2 & l \\ l & 1 \end{matrix} \right\|$$

from Eq. (2.1) we find the matrix

$$Q = \text{diag}\{2l\omega_0^2, 2l\}$$

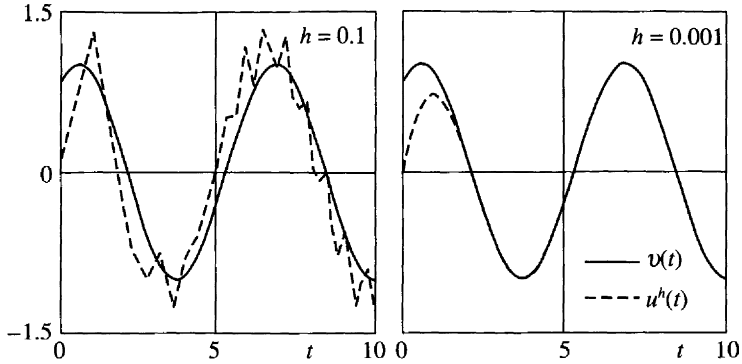

Fig. 1.

System (1.2) was adopted as the reference motion, i.e., in this case

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -\omega_0^2 x_1(t) - 2lx_2(t) + \upsilon(t), \quad \upsilon(t) = \sin(1+t)$$

with the initial state

$$x_1(0) = w_0, \quad x_2(0) = w_1$$

The elements $\xi_i^h = \left\{\xi_{1i}^h, \xi_{2i}^h\right\} \in R^2$ were calculated by means of the formulae

$$\xi_{1i}^h = y_1(\tau_i) + h\sin(1 + C\tau_i), \quad \xi_{2i}^h = y_2(\tau_i) + h\sin(1 + C\tau_i)$$

The control $u(t)$ in system (5.1) was found from formula (3.1), i.e.

$$u_i^h = \begin{cases} -K, & s_i \geq \alpha K \\ s_i\alpha^{-1}, & -\alpha K \leq s_i \leq \alpha K; \quad s_i = l(x_1(\tau_i) - \xi_{1i}^h) + (x_2(\tau_i) - \xi_{2i}^h) \\ K, & s_i \leq \alpha K \end{cases}$$

It was assumed that

$$L = 0.2, \quad \omega_0 = 1, \quad K = 2,$$

$$\delta_i(h) = dh(1+i)^{-\gamma}, \quad d = 10, \quad \gamma = 0.51, \quad C = 10$$

Figure 1 shows the evolution of the controls $\upsilon(t)$ and $u^h(t)$ when $\alpha = 0.5$.

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